Paper No.323/2021

APPLICATION OF A NUMERICAL-ANALYTICAL METHOD FOR SOLVING A ONE-DIMENSIONAL WAVE EQUATION

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Abstract: *The article contains a solution to the problem of wave propagation in a one-dimensional rod from the initial impact. A numerical-analytical method is used to solve the problem. The numerical part of the method is based on the application of the idea of the finite difference method. The analytical part uses the concept of Green's function to solve the problem in terms of the spatial coordinate in the considered area. The results include graphs of the solution obtained at different points in time.*

Keywords: *Distribution of waves, Euler's method, Green's function, finite difference method*

DOI: *10.36336/akustika20213922*

1. INTRODUCTION

The simplest method for the numerical solution of wave problems is the finite difference method [1], which allows one to solve equations in spaces of different dimensions. The most famous are explicit and implicit algorithms for solving. The use of explicit algorithms (one of the implementations of which is the Euler method), despite their simplicity, encounters a number of difficulties, one of which is rather strict restrictions on the sampling step in time. The application of the Euler method consists in sequential iterative calculation of the unknown values of the function at the nodes of the finite-difference spatial grid based on the previously calculated values of the function, but located at layers earlier in time. Implicit algorithms are more stable, but their application requires the solution of a system of equations, but presented the requirements for time steps in these algorithms are much softer. A more progressive, but slightly more resource-intensive, representative of numerical methods, which allows one to obtain a numerical solution of the wave problem, is the Euler method with recalculation („predictor-corrector"). The implementation of which presupposes the initial solution of the problem using the Euler method and subsequent refinement, that is, in fact, it requires not one, but two steps in one period of time. In the classical numerical procedure, there is a transition from the initial continuous problem to its finite-difference analogue, both spatial and temporal coordinates are discretized, that is, there is a transition to a finite system of algebraic equations with the desired values of the unknown function at the grid nodes. In both explicit and implicit solution schemes, it is required to perform sequential traversal of all temporal layers, from earlier to later layers. In contrast to numerical methods, the use of an analytical approach allows you to immediately obtain a solution at the right moment in time, but obtaining analytical solutions for areas with complex geometry is difficult throughout the space-time domain. Theoretically, the advantages of numerical and analytical methods can be combined using hybrid numerical-analytical methods for solving wave problems, where, for example, disctization in time is performed, while the solution is sought analytically in the coordinate. This article is devoted to one of the possible implementations of this approach.

2. MAIN PROVISIONS

The simplest equation describing wave processes in media is the one-dimensional wave equation, which is also often called the string vibration equation [2,3,4]. It allows one to describe the development of wave phenomena, in particular, the vibration of particles during mechanical deformation of an elastic medium. Its generalization to multidimensional cases allows one to describe waves in membranes and solids, the propagation of sound vibrations in liquids and gases, and also finds application for describing electromagnetic phenomena. A feature of the course of wave processes is the transfer of energy without transfer of the substance itself. In particular, when considering the vibration of a string, the unknown function entering into it describes the deviation of the points of the string from the equilibrium position. For linear wave phenomena, the sequence of the development of the process in time is characteristic, that is, the newly arrived state of the system is determined by the previous states, it is this property that is reflected in the application of numerical solution procedures. The most frequently used iterative algorithms are based on sequential consideration of individual states, where several time layers preceding the current one are connected by equations. Depending on the order of the finite-difference approximation of the derivatives, the number of connected space-time layers in the problem may differ, depending on the required solution accuracy. However, the number of simultaneously taken into account time layers cannot be less than two, which is determined by the order of the derivative of the function with respect to time entering into the wave equation.

In contrast to numerical procedures, when using analytical methods, the solution of the problem can be carried out immediately over the entire space-time domain. In particular, the application of the Green's function method for the wave equation makes it possible to find a solution to the problem for each moment of time. But within the framework of only an analytical approach, it can be difficult to take into account areas with complex geometry; therefore, the construction of a numerical-analytical procedure is highly desirable and will simplify the consideration of boundary conditions.

The article illustrates the possibility of using a numerical--analytical solution on the example of a one-dimensional wave equation. The solution plots illustrate the reflection of the wave from the boundaries of the computational domain, which consists in the amplification and attenuation of the wave amplitudes at different points of the domain at different times, as well as automatic satisfaction of the boundary conditions, that is, the problem of so-called „contour points" is removed. The Green's function itself is constructed taking into account the boundary conditions of the considered problem.

The perturbation of the string that arises at the initial moment of time sequentially propagates over time, captures more and more sections of the string, reaches the points of attachment with the specified boundary conditions, and, reflected from the boundaries of the region, again propagates inside the computational domain.

The article deals with the propagation of linear waves. A much more complex phenomenon is the propagation of nonlinear waves, while in the medium under the influence of the wave, the properties of the medium itself change, and this, in turn, changes the properties of the wave. In particular, the work [5] is devoted to the problem of propagation of nonlinear waves.

3. FORMULATION OF THE PROBLEM

3.1. Basic relations describing the propagation of linear waves in a string

As is known [2, 3], in the case of wave propagation in a string, only transverse waves are observed, while long waves are absent.

Let the function u (x, t) be the deviation of the string from the OX axis at the point x at the time t. Based on the d'Alembert principle, one can derive an equation for the vibrations of a string, which, according to [6], has the form:

$$diff(diff(u,t),t) = v^2 diff(diff(u,x),x) + p(x,t) \tag{1}$$

where

u is the deviation of individual points of the string, is the time,

diff (diff (u, t), t) is the double differentiation of the function of the deviation of the points of the string in time,

v^2 is the wave speed,

p (x, t) is some function of external influence.

The description of the derivation of this equation and the quantities included in it is given in the numerous educational literature on the equations of mathematical physics [4], [6].

Let us perform a finite-difference discretization of the problem in time, assuming that the derivative of the function with respect to the x coordinate can be found analytically and assuming the function ***p (t, x)*** to be equal to zero:

$$(u[i+1](x2) - 2\,u[i](x1) + u[i-1](x0))/ht2 = v^2 diff(diff(u[i+1],x2),x2) \tag{2}$$

where

u[i + 1](x) is the solution vector corresponding to the [i + 1] time layer,

u[i](x1) is the solution vector corresponding to the i-th time layer, and

u[i-1](x) - solution vector corresponding to [i-1] time layer.

Moving all the terms related to the layer [i + 1] to the left side of the equality, we get:

$$ht^2 v^2 diff(diff(u[i+1],x2),x2) - u[i+1](x2) = u[i-1](x0) - 2u[i](x1) \tag{3}$$

Assuming that the right-hand side is constant for the layer [i + 1], since it includes the vectors u[i-1](x0) -2u[i](x1) located on the previous time layers, we will solve this equation using the method Green's functions.

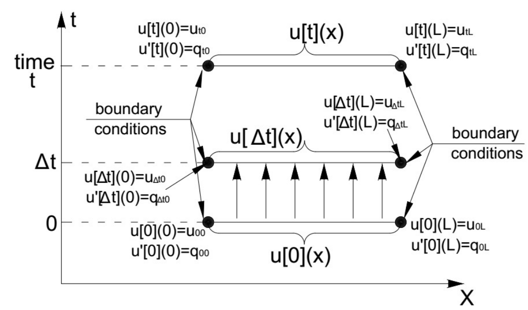

Fig. 1: Applying a time-step procedure to solving a time-dependent problem

The article assumes that the boundary conditions of the problem are permanent fixation at the ends of the considered spatial region and the boundary conditions are not subject to change over time. Then the expression for the Green's function for the linear expression ht^2v^2diff (diff (u [i + 1], x2), x2) -u [i + 1] (x2) will have the form:

$$G(x2,y) = -\left[0.5e^{-(x2+y)/ht/hv}(e^{2/ht/v} - e^{2x2/ht/v})(-1+e^{2y/ht/v})\theta(x2-y)\right]/ht/v - \left[0.5e^{-(x2+y)/ht/hv}(-1+e^{2x2/ht/v})(e^{2/ht/v} - e^{2/ht/v})\theta(-x2+y)\right]/(-1+e^{2/ht/v})/ht/v \tag{4}$$

where

x2 is the spatial coordinate corresponding to the [i + 1] time layer, and

y is the integration variable included in the integral representation of the solution to the differential equation,

ϑ(x) is the Heaviside function.

This expression is obtained using the Maxima system of symbolic-analytical calculations, taking into account a given differential operator and given boundary conditions, which, according to the conditions of the problem under consideration, do not depend on time. The appearance of a three-dimensional graph of the Green's function is shown in Fig.2:

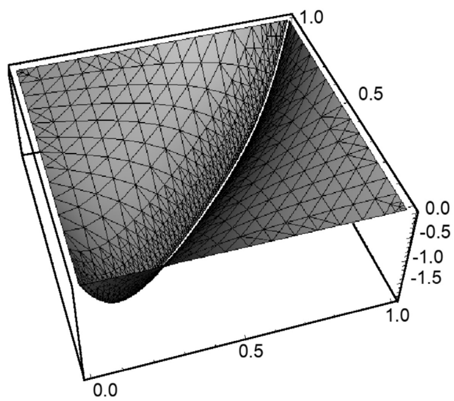

Fig. 2: 3D plot of Green's function on [i + 1] time layer

For a more detailed display of the topology of the Green's function, an isofield can be constructed, on which the trace of the Dirac delta function is clearly visible in the form of a white colorless stripe passing along the diagonal (Fig. 3)

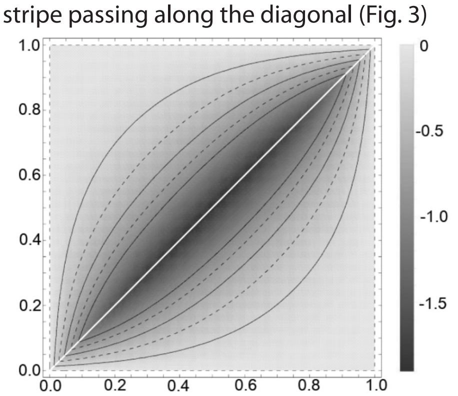

Fig. 3: Green's function isofield at [i + 1] time level

In the article, functions on two initial time layers are taken as the initial conditions. For the initial time period, the dependence 4 (0.25- (x-0.5) 2) is accepted, and for the time sampling step ht = 0.125 sec. moment, the dependence 4.4 (0.25- (x-0.5) 2) is adopted. The graphs of the initial conditions are shown in Fig. 4:

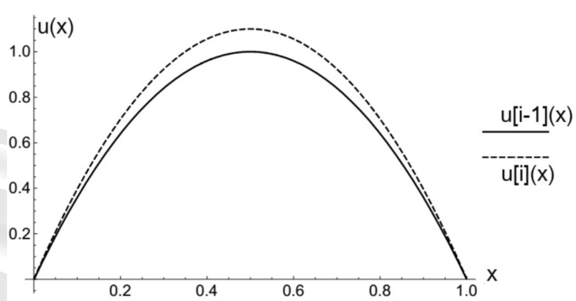

Fig. 4: Graphs of initial conditions on [i-1] and [i] time layer

As can be seen from the presented graphs, the boundary conditions in the coordinate are satisfied for both time layers. The expression for the derivative given at the initial moment of time can be found using the standard finite-difference approximation of the function:

$$\{u[1](x1) - u[0](x0)\}/ht = \{diff(u[t](x2), t), t = 0\} \quad (5)$$

where

u[1](x1) is the solution vector corresponding to the [1]-th time layer,

u[0](x0) is the solution vector corresponding to the [0]-th time layer, and

{diff (u[t](x2), t), t = 0} the value of the time derivative of the solution at the initial moment.

The solution to the differential equation can be represented as a convolution of the Green's function of the differential operator and the right-hand side of the equation (which plays the role of a constant expression, since the spatial coordinates x [i-1] and x [i] are located on the previous time layers[7],[8]:

$$u[i+1](x2) = \{u[i-1](x0) - 2u[i-1](x1)\}\, Int[G(x2, y), y = 0,1] \quad (6)$$

where

u[i + 1](x2) is the solution vector corresponding to the [i + 1] time layer,

u[i](x1) is the solution vector corresponding to the [i] -th time layer, and

u[i-1](x0) solution vector corresponding to [i-1] time layer,

Int (G (x2, y), y = 0,1) integral of Green's function on the interval from 0 to 1.

4. THE RESULTS OBTAINED FOR SOLVING A ONE-DIMENSIONAL WAVE PROBLEM BY A HYBRID NUMERICAL-ANALYTICAL METHOD

The graphs of the obtained solutions at different points in time are presented in the following Fig.5-8:

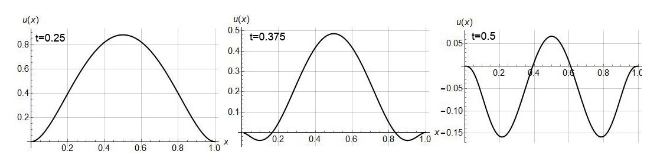

Fig. 5: Solution plots for t =0.25, t =0.375,t = 0.5 sec.

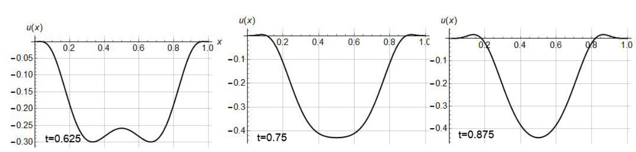

Fig. 6: Solution plots for t =0.625, t =0.75,t = 0.875 sec.

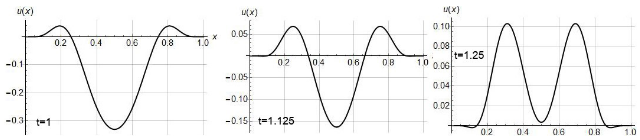

Fig. 7: Solution plots for t =1.0, t =1.125,t = 1.25 sec.

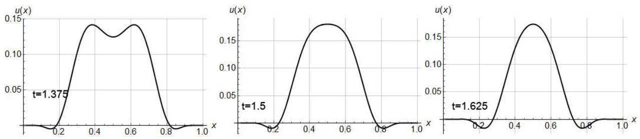

Fig. 8: Solution plots for t =1.375, t =1.5,t = 1.625 sec.

The presented solution graphs illustrate the possibility of obtaining a solution to the problem using the proposed numerical-analytical hybrid approach. Of course, it is possible to complicate the method by passing to an implicit scheme, or using the „predictor-corrector" method, but it is precisely the use of a simple iterative scheme of the Euler method that allows us to most clearly illustrate the technology of transition from a space-time problem to the sequential application of analytical solutions of spatial problems.

The graph and isofield of the solution to the problem in the space-time domain are presented in the following figure (Fig. 9):

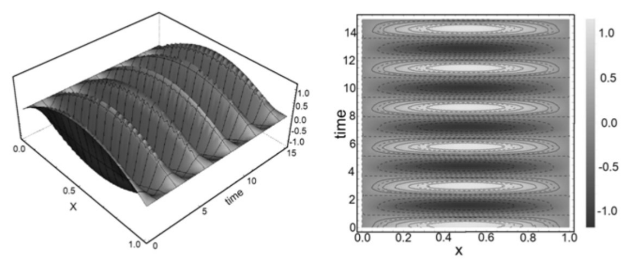

Fig. 9: Three-dimensional graph and isofield of the obtained solution in the space-time domain

Without a clue, the proposed procedure can be extended to the solution of problems of higher dimension, but in this case the method will inevitably face the problem of accumulating errors, since the newly calculated solutions depend on the previously calculated solutions at the previous time steps. Therefore, in the multidimensional case, in the presence of regions of complex shape, it will be necessary to calculate the desired function in a sufficiently large number of interior points of the region so that the found values can be used as initial values for the next time step.

5. CONCLUSION

An algorithm for the numerical-analytical solution of the wave problem for a one-dimensional wave differential equation is implemented. The application of the step-by-step iterative Euler method for approximating the problem in time is considered. At each belt step, the problem is solved by applying the Green's function. The obtained solutions illustrate the possibility of applying hybrid numerical-analytical procedures, which make it possible to separate the solution of spatio-temporal problems separately into solutions in spatial and temporal coordinates. The implementation of numerical-analytical algorithms was carried out using the programming language and the system of symbolic computations Maxima, graphs of the propagation of the wave process for the considered space-time domain were obtained.

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